

BASE CHANGE AND GROTHENDIECK DUALITY FOR COHEN-MACAULAY MAPS

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ABSTRACT. Let $f : X \rightarrow Y$ be a Cohen-Macaulay map of finite type between Noetherian schemes, and $g : Y' \rightarrow Y$ a base change map, with Y' Noetherian. Let $f' : X' \rightarrow Y'$ be the base change of f under g and $g' : X' \rightarrow X$ the base change of g under f . We show that there is a canonical isomorphism $\theta_g^f : g'^* \omega_f \simeq \omega_{f'}$, where ω_f and $\omega_{f'}$ are the relative dualizing sheaves. The map θ_f^g is easily described when f is proper, and has a subtler description when f is not. If f is *smooth* we show that θ_g^f corresponds to the canonical identification $g'^* \Omega_f^r = \Omega_{f'}^r$ of differential forms, where r is the relative dimension of f . Our results generalize the results of B. Conrad in two directions - we don't need the properness assumption, and we do not need to assume that Y and Y' carry dualizing complexes. Residual Complexes do not appear in this paper.

1. INTRODUCTION

Our approach to Grothendieck Duality is the approach of Deligne and Verdier [4], [20] with crucial elaborations by Alonso Tarrío, Jeremías Lopez and Lipman [1]. In particular, we do not use residual complexes or dualizing complexes—crucial ingredients in the approach laid out in Hartshorne's voluminous book [6]. Our intent is to show that the recent results of Conrad [2] on base change for duality can be attacked without recourse to dualizing or residual complexes, and this attack yields more general results in another direction—we don't need to assume that the our fundamental map of schemes (whose duality under base change we are examining) is *proper*—just of finite type. We, of course, do assume that this map is Cohen-Macaulay (as does Conrad). We also have results for base change for smooth maps—the primary motivation for Conrad's work—and in this case also our results do not assume properness (or the existence of residual complexes on the schemes involved).

Schemes will mean Noetherian schemes. For any scheme Z , Z_{qc} (resp. Z_c) will denote the category of quasi-coherent \mathcal{O}_Z -modules (resp. coherent \mathcal{O}_Z -modules). $D_{qc}^+(Z)$ will denote the derived category of bounded below quasi-coherent sheaves on Z .

In a short while we will give a quick summary of the Deligne-Verdier approach (DV approach for short) to Duality. The classic references are [4] and [20]. Deligne's and Verdier's results apply to (finite-type, separated) maps between schemes of finite Krull dimension. This is generalized to arbitrary schemes by Alonso Tarrío, Jeremías Lopez and Lipman in [1] (in fact their results are far more general than we need in this paper. They work with formal schemes). Since our interest is not

restricted to schemes with *finite* Krull dimension, we will appeal to [1] for our results (and make a respectful bow towards [4] and [20] by also giving appropriate references to the analogous results there). The key results in the DV approach to Duality are (a) the existence of a right adjoint to the (derived) direct image functor for a proper map—the *twisted inverse image functor* in Verdier’s terminology [4, pp. 416–417], [20, pp. 393–394, Theorem 1] and [1, p. 5, Theorem 1], (b) compatibility of this twisted inverse image functor with *flat* base change [20, pp. 394–395, Theorem 2], [1, pp. 8–9, Theorem 3] and as a consequence (c) the *localness* of the twisted inverse image functor [20, p. 395, Corollary 1], [1, p. 88, Proposition 8.3.1]. We should point out that Neeman has an intriguingly different approach to the above results (see [19]).

Here then is promised summary of the key points of the DV-approach. Let $f : X \rightarrow Y$ be a separated map of schemes. One “constructs” a functor $f^! : D_{qc}^+(Y) \rightarrow D_{qc}^+(X)$ in two steps. If f is proper, then $f^!$ is defined as the right adjoint to $Rf_* : D_{qc}^+(X) \rightarrow D_{qc}^+(Y)$ (which is shown to exist—see (a) above). If f is not proper, then we pick a compactification $\bar{f} : \bar{X} \rightarrow Y$ of f , and $f^!$ is defined to be the restriction of $\bar{f}^!$ to $D_{qc}^+(X)$. The local nature of $f^!$ (see (c) above. We say more about this in Remark 1.0.1 below) ensures that the end product is independent of the compactification \bar{f} . Recall that Nagata’s result in [18] ensures the existence of a compactification of f . Recently there have been other proofs of Nagata’s result by Lütkebohmert [16] and independently Conrad [3].

Remark 1.0.1. Here is how the local nature of “upper shriek” is proved using flat base change. Suppose first that we have two compactifications $(\iota_k, f_k : X_k \rightarrow Y)$ of f and these compactifications can be embedded in a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\iota_1} & X_1 & & \\ \parallel & & \downarrow h & \searrow f_1 & \\ X & \xrightarrow{\iota_2} & X_2 & \xrightarrow{f_2} & Y \end{array}$$

with the square being cartesian. Since $f_1^! \simeq h^! f_2^!$ therefore by flat base change we have

$$\iota_1^* f_1^! \xrightarrow{\sim} \iota_1^* h^! f_2^! \xrightarrow{\sim} Id_X^! \iota_2^* f_2^! = \iota_2^* f_2^!.$$

The point is that this isomorphism has another description which is more useful at times. Let $T_h : Rh_* f_1^! \rightarrow f_2^!$ be the map that arises from the isomorphism $f_1^! \simeq h^! f_2^!$. Then the above isomorphism can be described by

$$\iota_1^* f_1^! = \iota_2^* Rh_* f_1^! \xrightarrow{\iota_2^* T_h} \iota_2^* f_2^!. \quad (1.1)$$

This latter description is used in the proof of Proposition 3.1.1 and in Proposition 4.3.1.

We have assumed that f_1 and f_2 are related by diagrams of the form above. The general case can be reduced to this by considering the closure of the diagonal embedding $X \rightarrow X_1 \times_Y X_2$. If $\mu_{ij} : \iota_j^* f_j^! \xrightarrow{\sim} \iota_i^* f_i^!$ is the isomorphism described above for two compactifications (ι_i, f_i) and (ι_j, f_j) of f , then it is easy to see that

- μ_{ij} is compatible with open immersions into X .
- For three compactifications, we have

$$\mu_{ij} \circ \mu_{jk} = \mu_{ik}.$$

We should point out that in a different context (but with the same formalism) this has been worked out by Lipman in [13, p.46, Lemma (4.6)].

1.1. The Problem: To explain the problem, we will first consider a simpler situation, in which we have more hypotheses than we really need. With Conrad consider first a proper map $f : X \rightarrow Y$ of finite type between Noetherian schemes, which is *Cohen-Macaulay of relative dimension r* . The condition in italics means

- f is flat (of finite type) and;
- the non-empty fibers of f are Cohen-Macaulay of pure dimension r .

It is well-known that in this situation

$$f^! \mathcal{O}_Y \xrightarrow{\sim} \omega_f[r]$$

for some coherent sheaf ω_f (the relative dualizing sheaf) on X [12, p. 39, Lemma 1(i)]. It is further proved in *loc.cit.* that ω_f is flat over Y . It should be pointed out that the statement in *loc.cit.* is for $r = 2$, but the proof works for arbitrary r .

Let

$$\int_f = \int_f^{\mathcal{O}_Y} : R^r f_* \omega_f \rightarrow \mathcal{O}_Y$$

be induced by the trace map $T_f : Rf_* f^! \mathcal{O}_Y \rightarrow \mathcal{O}_Y$. The pair (ω_f, \int_f) induces a functorial isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(R^r f_* \omega_f, \mathcal{O}_Y)$ for $\mathcal{F} \in X_{qc}$, and hence (ω_f, \int_f) is unique up to unique isomorphism. Next suppose f is embedded in a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Since $R^r f_* \omega_f \otimes f^*(-)$ and $R^r f'_* \omega_{f'} \otimes (-)$ are both right exact functors, and since tensor products, pullbacks, and higher direct images commute with direct limits, we see that the natural map $g^* R^r f_* \omega_f \rightarrow R^r f'_* g'^* \omega_f$ is an isomorphism. We have, therefore, a map

$$g^\# \int_f : R^r f'_* g'^* \omega_f \rightarrow \mathcal{O}_{Y'}$$

defined by the composition

$$R^r f'_* g'^* \omega_f \xrightarrow{\sim} g^* R^r f_* \omega_f \xrightarrow{g^* \int_f} g^* \mathcal{O}_Y = \mathcal{O}_{Y'}.$$

The universal property of $(\omega_{f'}, \int_{f'})$ (note that f' is also Cohen-Macaulay of relative dimension r) immediately gives us a (unique) map

$$\theta_g^f : g'^* \omega_f \longrightarrow \omega_{f'}$$

such that $\int_{f'} \circ R^r f'_*(\theta_g^f) = g^\# \int_f$. Conrad's main results, when Y and Y' carry dualizing complexes, are

1. θ_g^f is an isomorphism.
2. If f is *smooth*, so that (via Verdier's identification [20, p.397, Theorem 3]) $\omega_f = \Omega_f^r$, $\omega_{f'} = \Omega_{f'}^r$, then θ_g^f is the canonical identification of differential forms $g'^* \Omega_f^r = \Omega_{f'}^r$. Here Ω_f^r and $\Omega_{f'}^r$ are the respective relative Kähler r -forms on X and X' .

A natural question is—how necessary is the hypothesis of properness for this result? On the face of it—extremely necessary, for the very definition of θ_g^f needs f to be proper. But, perhaps we are not being imaginative enough. Suppose we drop the properness assumption of f . Then f can (at least locally) be compactified by $\bar{X} \xrightarrow{\bar{f}} Y$ whose fibers are equidimensional (of pure dimension r). These compactifications need not be Cohen-Macaulay, but if we set $\omega_{\bar{f}} = H^{-r}(\bar{f}^! \mathcal{O}_Y)$, then we have a functorial isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_{\bar{f}}) = \mathrm{Hom}_{\mathcal{O}_Y}(R^r \bar{f}_* \mathcal{F}, \mathcal{O}_Y)$ for $\mathcal{F} \in X_{qc}$, whence an “integral” $\int_{\bar{f}} : R^r \bar{f}_* \omega_{\bar{f}} \rightarrow \mathcal{O}_Y$. Arguing as before, we get a map

$$\theta_{\bar{g}}^{\bar{f}} : \bar{g}'^* \omega_{\bar{f}} \rightarrow \omega_{\bar{f}'}$$

where $\bar{f}' : \bar{X}' \rightarrow Y'$ is the base change of f under g , and $\bar{g} : \bar{X}' \rightarrow \bar{X}$ the base change of g under \bar{f} . $\theta_{\bar{g}}^{\bar{f}}$ need not be an isomorphism (see [10, p. 773, Remark 3.4]), but we ask:

- (a) Is $\theta_g^f := \theta_g^{\bar{f}}|_{X'} : g^* \omega_f \rightarrow \omega_{f'}$ an isomorphism?
- (b) Is θ_g^f independent of the compactification of \bar{f} ?
- (c) Finally, if f is smooth, is θ_g^f the canonical identification of differential forms?

That then is the problem. In this paper, we answer all three questions affirmatively. In particular if $\{U_\alpha \xrightarrow{f_\alpha} Y\}$ is an open cover of $X \xrightarrow{f} Y$ such that each f_α has an equidimensional compactification, the various $\theta_{g_\alpha}^{f_\alpha}$ patch together on X' to give a global isomorphism $\theta_g^f : g^* \omega_f \xrightarrow{\sim} \omega_{f'}$, which is obviously independent of the cover $\{U_\alpha\}$. On the smooth locus of f , the above isomorphism is the canonical identification of differential forms. We state our results precisely in Theorem 2.2.1 and Theorem 2.2.2.

Our techniques are such that we do not need dualizing complexes or their Cousin versions—residual complexes. The author confesses to having a soft corner for the DV approach. He has often felt that the existence of $f^!$ for proper f and the flat base change theorem should be used to rebuild duality despite admonitions that there is “no royal road”. Here, for what it is worth, is our idea of the first mile of the royal road. In later work (with S. Nayak), we hope to use this approach to rework the theory of residues of Kunz, Hubl, Lipman [7], [8], [9], [13], [14].

Remark 1.1.1. We have quoted Lemma 1, p. 39 of Lipman’s paper [12] for a proof of the fact that the relative dualizing complex is concentrated in one degree, and the corresponding homology is flat over the base. The same Lemma also asserts that the ω_f is well behaved with respect to base change, but this assertion is not completely proved there. The proof given in *loc.cit.* shows that there are local isomorphisms between $g'^* \omega_f$ and $\omega_{f'}$, but it is not clear that these isomorphisms patch.

2. THE MAIN RESULTS

2.1. Verdier’s isomorphism: Let $f : X \rightarrow Y$ be a *smooth* separated map of finite type. Theorem 3 (p. 397) in Verdier’s paper [20] gives an isomorphism

$$f^! \mathcal{O}_Y \xrightarrow{\sim} \Omega_f^r[r] \tag{2.1}$$

for f *smooth*. We give Verdier’s proof in section 7, subsection 7.1. The theorem depends only upon his flat base change theorem [*loc.cit.*, Theorem 2]. In view of the results of [1] we do not have to assume that the schemes involved are of finite Krull

dimension. Moreover, from the proof of *loc.cit.* it is clear that the isomorphism (2.1) localizes well over open sets in X . This has implications when f is just smooth and of finite type (not necessarily separated).

2.2. Kleiman's functor f^K : For $f : X \rightarrow Y$ equidimensional of dimension r (i.e. f is of finite type, dominant, and its non-empty fibers have pure dimension r), consider the following variant of Kleiman's r -dualizing functor, $f^K = H^{-r}(f^!): Y_{qc} \rightarrow X_{qc}$ (see [11] for the definition of an r -dualizing functor).¹ For f *proper* we claim that f^K is indeed Kleiman's r -dualizing functor. Indeed, under our hypotheses on f , $H^{-k}(f^!\mathcal{G}) = 0$ for $k > r$, $\mathcal{G} \in Y_{qc} \subset D_{qc}^+(Y)$. Therefore we have a bifunctorial isomorphism (from the adjoint relationship between $f^!$ and Rf_*)

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^K \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(R^r f_* \mathcal{F}, \mathcal{G})$$

for $\mathcal{F} \in X_{qc}$ and $\mathcal{G} \in Y_{qc}$. The adjoint relationship between f^K and $R^r f_*$ immediately gives rise to an \mathcal{O}_Y -linear *integral*

$$\int_f^{\mathcal{G}} : R^r f_* f^K \mathcal{G} \longrightarrow \mathcal{G}.$$

The pair $(f^K \mathcal{G}, \int_f^{\mathcal{G}})$ is unique up to unique isomorphism.

Note that $(-)^K$ is local (in the sense that $(-)^!$ is local. See Remark 1.0.1). In fact the local property of $(-)^K$ follows from the local property of $(-)^!$. This gives another approach to the rather tedious proofs in [10, pp.753–754, Theorem 1.1], and [ibid, pp.776–777, Remark 3.7], though it must be pointed out that in *ibid* derived categories were eschewed and our hands were tied by the fact that we had to control the fiber dimensions of f . The local nature of $(-)^K$ means, among other things, that $(-)^K \mathcal{G}$ forms a sheaf on the (big) Zariski site over Y (consisting of equidimensional finite type schemes over Y). If $\iota : U \rightarrow X$ is an open immersion, and $f_U : U \rightarrow Y$ the map induced by the $f : X \rightarrow Y$ as above, then

$$\beta_{\iota} = \beta_{\iota}(f) : \iota^* f^K \xrightarrow{\sim} f_U^K \quad (2.2)$$

will denote the resulting functorial isomorphism.

Remark 2.2.1. In view of the remarks made towards the end of subsection 2.1, it is clear that if f is smooth and not necessarily separated, we have an isomorphism

$$f^K \mathcal{O}_Y \xrightarrow{\sim} \Omega_f^r. \quad (2.3)$$

Remark 2.2.2. For $\mathcal{G} \in Y_{qc}$, since the complex $f^! \mathcal{G}$ has no cohomology below $-r$, therefore we have a natural map in $D_{qc}^+(X)$, $\kappa_{\mathcal{G}} : f^K \mathcal{G}[r] \rightarrow f^! \mathcal{G}$. Next, since $R^k f_* f^K \mathcal{G} = 0$ for $k > r$, therefore we have a map $\kappa'_{\mathcal{G}} : Rf_* f^K \mathcal{G}[r] \rightarrow R^r f_* f^K \mathcal{G}$. One checks easily that

$$T_f(\mathcal{G}) \circ Rf_*(\kappa_{\mathcal{G}}) = \int_f^{\mathcal{G}} \circ \kappa'_{\mathcal{G}}.$$

¹Note that, since we are not assuming separatedness now, $f^!$ does not make sense. However, its $-r$ th cohomology does make sense. To begin with, X can be covered by open subschemes on which $f^!$ is defined. Over triple intersections, these objects formally satisfy cocycle rules. But that is not enough to glue them together as objects in $D_{qc}^+(X)$ (the reason why Hartshorne upgrades his constructions to Cousin complexes). However, the $-r$ th cohomology does glue together since we are now in the category of sheaves! This is the slick way of understanding [10, p. 760, Corollary 1.7].

Set $\omega_f = f^K \mathcal{O}_Y$. For f proper, if no confusion arises, we will write \int_f for $\int_f^{\mathcal{O}_Y}$. The pair (ω_f, \int_f) is called a *dualizing pair* for f . Now suppose we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array} \quad (2.4)$$

with f proper and as above. The canonical map $g^* R^r f_* \rightarrow R^r f'_* g'^*$ is an isomorphism (as before, the argument involves the fact that the above map is local in Y and Y' , $R^r f_* \mathcal{F} \otimes_{\mathcal{O}_Y} (-)$, $R^r f'_* \mathcal{F} \otimes_{\mathcal{O}_{X'}} f'^*(-)$ are right exact, and finally the fact that tensor products, pull-backs and higher direct images all commute with direct limits). Hence—as in the Cohen-Macaulay case—we have a map

$$g^\# \int_f : R^r f'_* g'^* \omega_f \rightarrow \mathcal{O}_{Y'}$$

induced by $g^* \int_f$ and the above isomorphism of functors. As before we have a map

$$\theta_g^f : g'^* \omega_f \rightarrow \omega_{f'}. \quad (2.5)$$

Our main theorem is:

Theorem 2.2.1. *Let*

$$\begin{array}{ccccc} \bar{X}' & & \xrightarrow{\bar{g}} & & \bar{X} \\ & \swarrow \bar{f}' & & \searrow \bar{f} & \\ & X' & \xrightarrow{g'} & X & \\ & \downarrow f' & & \downarrow f & \\ & Y' & \xrightarrow{g} & Y & \end{array}$$

be a commutative diagram of schemes such that

- f is Cohen-Macaulay of relative dimension r ;
- ι is an open immersion;
- \bar{f} is proper and equidimensional of dimension r ;
- the inner square, the outer trapezium, and the trapezium bordered by g' , ι' , ι and \bar{g} are all cartesian.

Then

- the map $\theta_g^f|_{X'} : g'^* \omega_f \rightarrow \omega_{f'}$ is independent of the compactification \bar{f} of f . Call the map θ_g^f .
- θ_g^f is an isomorphism.
- If f is smooth, and we identify $\omega_f, \omega_{f'}$ respectively with $\Omega_f^r, \Omega_{f'}^r$, via Verdier's isomorphism (2.1) (or (2.3)), then θ_g^f is the canonical identification of differential forms $g'^* \Omega_f^r = \Omega_{f'}^r$.

Explanation: Item (a) needs slight elaboration. Suppose $\bar{f}_j : \bar{X}_j \rightarrow Y$, $j = 1, 2$, are two equidimensional compactifications of f , with $\iota_j : X \rightarrow \bar{X}_j$ the corresponding open immersion. Suppose (in an obvious notation) $\beta_j : \omega_f \rightarrow \iota_j^* \omega_{\bar{f}_j}$ and $\beta'_j : \omega_{f'} \rightarrow$

$\iota_j^* \omega_{\tilde{f}_j}$, $j = 1, 2$ are the resulting isomorphisms (see equation (2.2)) Then (a) asserts that

$$\beta_1'^{-1} \circ \iota_1'^* \theta_g^{\tilde{f}_1} \circ \beta_1 = \beta_2'^{-1} \circ \iota_2'^* \theta_g^{\tilde{f}_2} \circ \beta_2.$$

Now suppose $f : X \rightarrow Y$ is Cohen-Macaulay of relative dimension r (not necessarily separated) and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}.$$

For each point $x \in X$, closed in its fiber, we can find an open neighborhood U of x and a quasi-finite map $h_U : U \rightarrow \mathbb{P}_Y^r$ such that $f|_U = \pi_Y \circ h_U$ ($\pi_Y =$ the projection map $\mathbb{P}_Y^r \rightarrow Y$). By *Zariski's Main Theorem*, h_U can be compactified by a finite map $\bar{h}_U : \bar{X}_U \rightarrow \mathbb{P}_Y^r$. \bar{X}_U is equidimensional and proper over Y . In other words, X can be covered by open subsets $\{U_\alpha\}$ such that each map $f_\alpha := f|_{U_\alpha} : U_\alpha \rightarrow Y$ can be compactified by an equidimensional map \bar{f}_α . By part (a) of the previous theorem, the maps $\theta_g^{f_\alpha}$ glue together to give a global $\mathcal{O}_{X'}$ -map

$$\theta_g^f : g'^* \omega_f \longrightarrow \omega_{f'}.$$

This map (again from part (a) of Theorem 2.2.1) is independent of the cover $\{U_\alpha\}$. Part (b) of the theorem then implies that θ_g^f is an isomorphism. Therefore Theorem 2.2.1 has the following, seemingly more general reformulation.

Theorem 2.2.2. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of schemes, with f Cohen-Macaulay of relative dimension r . Then,

(a) *there exists an isomorphism*

$$\theta_g^f : g'^* \omega_f \xrightarrow{\sim} \omega_{f'}$$

characterized by the property that if $U \subset X$ is an open set admitting an equidimensional compactification over Y and $\theta_g^{f|_U}$ is the map in Theorem 2.2.1(a), then

$$\theta_g^f|_{g'^{-1}(U)} = \theta_g^{f|_U}.$$

(b) *If f is smooth and $\omega_f, \omega_{f'}$ are identified with $\Omega_f^r, \Omega_{f'}^r$ via (2.3), then θ_g^f is the canonical identification of differential forms $g'^* \Omega_f^r = \Omega_{f'}^r$.*

Remark 2.2.3. Let $f : X \rightarrow Y$ be proper and equidimensional of dimension r and consider the base change diagram (2.4). Suppose $(\tilde{\omega}_f, \tilde{f}_f)$ (resp. $(\tilde{\omega}_{f'}, \tilde{f}_{f'})$) was another dualizing pair for f (resp. f'). Let $\tilde{\theta}_g^f : g'^* \tilde{\omega}_f \rightarrow \tilde{\omega}_{f'}$ be the map defined in

the θ_g^f was defined in (2.5). Then

$$\begin{array}{ccc} g'^* \omega_f & \xrightarrow{\theta_g^f} & \omega_{f'} \\ \uparrow \simeq & & \uparrow \simeq \\ g'^* \tilde{\omega}_f & \xrightarrow{\tilde{\theta}_g^f} & \tilde{\omega}_{f'} \end{array}$$

commutes, where the vertical isomorphisms arise from uniqueness (up to unique isomorphism) of dualizing pairs. Indeed, if $\eta : \tilde{\omega}_f \rightarrow \omega_f$ and $\eta' : \tilde{\omega}_{f'} \rightarrow \omega_{f'}$ are these unique isomorphisms, then

$$\int_{f'} \circ R^r f'_* (\theta_g^f \circ g'^* \eta) = \int_{f'} \circ R^r f'_* (\theta_g^f) \circ R^r f'_* (g'^* \eta) = g^\# \int_f \circ R^r f'_* (g'^* \eta) = g^\# \int_f$$

and

$$\int_{f'} \circ R^r f'_* (\eta' \circ \tilde{\theta}_g^f) = \int_{f'} \circ R^r f'_* (\eta') \circ R^r f'_* (\tilde{\theta}_g^f) = \tilde{\int}_{f'} \circ R^r f'_* (\tilde{\theta}_g^f) = g^\# \tilde{\int}_f.$$

Thus by the universal property of $(\omega_{f'}, \int_{f'})$,

$$\eta' \circ \tilde{\theta}_g^f = \theta_g^f \circ g'^* \eta.$$

Note, in particular, that θ_g^f is an isomorphism if and only if $\tilde{\theta}_g^f$ is.

Remark 2.2.4. Suppose f, g are as in the preceding remark and the diagram (2.4) can be embedded in a larger commutative diagram

$$\begin{array}{ccccc} S' & & \xrightarrow{h'} & & S \\ & \searrow v' & & \swarrow v & \\ & X' & \xrightarrow{g'} & X & \\ & \downarrow f' & & \downarrow f & \\ & Y' & \xrightarrow{g} & Y & \\ & \swarrow u' & & \searrow u & \\ T' & & \xrightarrow{h} & & T \end{array}$$

in which the inner square, outer square, and the two trapeziums squeezed between them are all cartesian. Then one checks easily that the following diagram commutes

$$\begin{array}{ccccc} h'^* v^* \omega_f & \xrightarrow{h'^* \theta_u^f} & h'^* \omega_{f_T} & \xrightarrow{\theta_h^{f_T}} & \omega_{f'_T} \\ \parallel & & & & \parallel \\ v'^* g'^* \omega_f & \xrightarrow{v'^* \theta_g^f} & v'^* \omega_{f'} & \xrightarrow{\theta_{u'}^{f'}} & \omega_{f'_T} \end{array}$$

The strategy is as follows. Let $\varphi_1 = \theta_h^{f_T} \circ h'^* \theta_u^f$ and $\varphi_2 = \theta_{u'}^{f'} \circ v'^* \theta_g^f$. Then one checks (using the definitions of the various θ 's) that

$$\int_{f'_T} \circ R^r f'_{T*}(\varphi_1) = \int_{f'_T} \circ R^r f'_{T*}(\varphi_2).$$

This implies, by the dualizing property of $(\omega_{f'_T}, \int_{f'_T})$, that $\varphi_1 = \varphi_2$.

3. MAIN IDEAS

The key idea is this—one defines a residue map $\text{res}_Z : R_Z^r f_* \omega_f \rightarrow \mathcal{O}_Y$ for appropriate closed subschemes Z of X . The residue map is a formal analogue of the integral \int_f . One shows that this residue map for special Z (we call such Z 's *good*) has a local duality property and is well behaved with respect to base change. Recall that $Z \xrightarrow{j} X$ is a closed subscheme of X , then $R_Z^p f_*$ is the p th derived right derived functor of $f_* \Gamma_Z$. The corresponding derived functor $D_{qc}^+(X) \rightarrow D_{qc}^+(Y)$ is denoted $R_Z f_*$.

3.1. Residues: Let $f : X \rightarrow Y$ be a separated Cohen-Macaulay map, $Z \xrightarrow{j} X$ a closed immersion such that $h = j \circ f : Z \rightarrow Y$ is *proper*. Suppose

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \bar{X} \\ & \searrow f & \downarrow \bar{f} \\ & & Y \end{array}$$

is a compactification of f . In D_{qc}^+ we have a sequence of maps

$$R_Z f_* \omega_f[r] \xrightarrow{\sim} R_Z f_* \iota^* \bar{f}^! \mathcal{O}_Y \xrightarrow{\sim} R_Z \bar{f}_* \bar{f}^! \mathcal{O}_Y \longrightarrow R \bar{f}_* \bar{f}^! \mathcal{O}_Y \xrightarrow{T_f} \mathcal{O}_Y.$$

Taking the 0-th cohomology of the above composition we get the $(\mathcal{O}_Y$ -linear) *residue map*:

$$\text{res}_Z : R_Z^r f_* \omega_f \longrightarrow \mathcal{O}_Y. \quad (3.1)$$

Proposition 3.1.1. *The residue map $\text{res}_Z : R_Z^r f_* \omega_f \rightarrow \mathcal{O}_Y$ does not depend on the compactification (ι, \bar{f}) of f .*

Proof. Let $(\iota_k, f_k : X_k \rightarrow Y)$, $k = 1, 2$ be two compactifications of f . By taking the closure of the diagonal embedding of X in $X_1 \times_Y X_2$ if necessary, we may assume that we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\iota_1} & X_1 & & \\ \parallel & & \downarrow h & \searrow f_1 & \\ X & \xrightarrow{\iota_2} & X_2 & \xrightarrow{f_2} & Y \end{array}$$

with the square being cartesian. The Proposition follows from the commutativity of

$$\begin{array}{ccccccc}
& & R_Z f_* \iota_1^* f_1^! \mathcal{O}_Y & \xrightarrow{\sim} & R_Z f_{1*} f_1^! \mathcal{O}_Y & \longrightarrow & R f_{1*} f_1^! \mathcal{O}_Y \\
& \nearrow \sim & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
R_Z f_* \omega_f[r] & & R_Z f_* \iota_2^* R h_* f_1^! \mathcal{O}_Y & \xrightarrow{\sim} & R_Z f_{2*} R h_* f_1^! \mathcal{O}_Y & \longrightarrow & R f_{2*} R h_* f_1^! \mathcal{O}_Y \\
& \searrow \sim & \downarrow T_h & & \downarrow T_h & & \downarrow T_h \\
& & R_Z f_* \iota_2^* f_2^! \mathcal{O}_Y & \xrightarrow{\sim} & R_Z f_{2*} f_2^! \mathcal{O}_Y & \longrightarrow & R f_{2*} f_2^! \mathcal{O}_Y
\end{array}$$

$\xrightarrow{T_{f_1}}$
 $\xrightarrow{T_{f_2}}$

We point out that the triangle on the left commutes since it does so before applying the functor $R_Z f_*$ (see Remark 1.0.1, especially the isomorphism (1.1)). \square

Remark 3.1.1. If $(\iota, \bar{f} : \bar{X} \rightarrow Y)$ is a compactification of f such that \bar{f} is equidimensional of dimension r (so that $\bar{f}^K : Y_{qc} \rightarrow \bar{X}_{qc}$ is defined), then with $\omega_{\bar{f}} = \bar{f}^K \mathcal{O}_Y$ we see easily that res_Z can be defined by the commutativity of

$$\begin{array}{ccccc}
R_Z^r f_* \omega_f & \xrightarrow{\sim} & R_Z^r f_* \iota^* \omega_{\bar{f}} & \xrightarrow{\sim} & R_Z^r \bar{f}_* \omega_{\bar{f}} \\
\text{res}_Z \downarrow & & & & \downarrow \\
\mathcal{O}_Y & \xleftarrow{\int_{\bar{f}}} & R^r \bar{f}_* \omega_{\bar{f}} & &
\end{array}$$

We are not interested in arbitrary Z . Our interest is in “good” immersions, which we now define:

Definition 3.1.1. Let f, Z be as above. $j : Z \hookrightarrow X$ is said to be *good* if it satisfies the following hypotheses (cf. also [8, (4.3)]):

- There is an open subscheme $V \subset X$, *affine over* Y , such that $j : Z \hookrightarrow X$ factors through V .
- There is an affine open covering $\{U_\alpha = \text{Spec } A_\alpha\}$ of Y such that if $V_\alpha = \text{Spec } R_\alpha$ is the inverse image of U_α in V (under f), then the closed immersion j is given in V_α by a regular R_α -sequence.
- $h = f \circ j : Z \rightarrow Y$ is *finite*.

3.2. Three key propositions: For proving Theorem 2.2.1 parts (a) and (b) the crucial ingredients are (a) Local Duality (Proposition 3.2.1 below), (b) Compatibility of Local Duality with base change (Proposition 3.2.2 below); and (c) Compatibility between the base change isomorphism for $R_Z^r f_*$ and $R^r f_*$ (Proposition 3.2.3 below). The proofs of these key propositions will be given later, after we show (in subsection 3.3 how a substantial part of the main result Theorem 2.2.1 is proved using these propositions

For good immersions $Z \xrightarrow{j} X$ we have the following version of local duality. Let \hat{X} be the formal scheme obtained by completing X along Z , and $\hat{f} : \hat{X} \rightarrow Y$ the resulting morphism. Let \hat{X}_{qc} denote the category of quasi coherent $\mathcal{O}_{\hat{X}}$ -modules. The functor $R_Z^r f_* : X_c \rightarrow Y_{qc}$ “extends” to a functor $R_Z^r \hat{f}_* : \hat{X}_c \rightarrow Y_{qc}$ in a natural way. For any coherent sheaf \mathcal{F} defined in an open neighborhood of Z in X , let $\hat{\mathcal{F}}_Z$ denote the completion of \mathcal{F} along Z . Let D_Z be the functor on \hat{X}_c given by $D_Z = \text{Hom}_{\mathcal{O}_Z}(R_Z^r \hat{f}_*(\cdot), \mathcal{O}_Y)$. Making the identification $R_Z^r \hat{f}_* \widehat{\omega_{f,Z}} = R_Z^r f_* \omega_f$, we have the following Local Duality assertion:

Proposition 3.2.1. (Local Duality) *The pair $(\widehat{\omega_{f,Z}}, \text{res}_Z)$ represents D_Z .*

Next suppose

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a cartesian square (f as before, Cohen-Macaulay of relative dimension r and separated). Suppose $j : Z \hookrightarrow X$ is a good immersion for f . Let $j' : Z' \hookrightarrow X'$ and $g'' : Z' \rightarrow Z$ be the corresponding base change maps. Note that $j' : Z' \hookrightarrow X'$ is a good immersion for f' . Define

$$g^\# \text{res}_Z : R_{Z'}^r f'_* g'^* \omega_f \longrightarrow \mathcal{O}_Y$$

by the commutativity of

$$\begin{array}{ccc} g^* R_Z^r f_* \omega_f & \xrightarrow{\sim} & R_{Z'}^r f'_* g'^* \omega_f \\ & \searrow g^* \text{res}_Z & \downarrow g^\# \text{res}_Z \\ & & \mathcal{O}_{Y'} \end{array}$$

The horizontal isomorphism is the canonical base change isomorphism (defined for e.g. in (5.3)). We then have:

Proposition 3.2.2. *The pair $(\widehat{g'^* \omega_{f,Z'}}, g^\# \text{res}_Z)$ represents $D_{Z'}$.*

With notations as above, we have:

Proposition 3.2.3. *The following diagram commutes:*

$$\begin{array}{ccc} g^* R_Z^r f_* \omega_f & \longrightarrow & g^* R^r \bar{f}_* \omega_{\bar{f}} \\ \simeq \downarrow & & \downarrow \simeq \\ R_{Z'}^r f'_* g'^* \omega_f & \longrightarrow & R^r \bar{f}_* \bar{g}^* \omega_{\bar{f}} \end{array}$$

where the vertical arrows are the canonical base change isomorphisms.

3.3. Proof of Theorem 2.2.1 (a) and (b): Consider the situation in Theorem 2.2.1. First assume that we have a good immersion $j : Z \hookrightarrow X$ for f (a strong assumption, and in general there is no guarantee that such an immersion exists). Let $Z' = g'^{-1}(Z)$. Then as consequence of Proposition 3.2.3 above and the definition of $g^\# \text{res}_Z$, we see that

$$g^\# \text{res}_Z = \text{res}_{Z'} \circ R_{Z'}^r f'_* (\theta_{g'}^{\bar{f}}).$$

Since $(\widehat{g'^* \omega_{f,Z'}}, g^\# \text{res}_Z)$ and $(\widehat{\omega_{f',Z'}}, \text{res}_{Z'})$ represent the same functor it follows that

- With obvious notation, $\widehat{\theta_{g,Z'}^{\bar{f}}}$ does not depend on the compactification \bar{f} . Indeed res_Z , $g^\# \text{res}_Z$, and $\text{res}_{Z'}$ are all independent of \bar{f} giving the conclusion.
- $\widehat{\theta_{g,Z'}^{\bar{f}}}$ is an isomorphism.

As a consequence, for every $x' \in Z'$, we have

- (a) $\theta_{g,x'}^{\bar{f}}$ does not depend on \bar{f} ;
- (b) $\theta_{g,x'}^{\bar{f}}$ is an isomorphism.

The difficulty is in finding enough good immersions in X . This is where the Cohen-Macaulay property helps. In the flat topology on X we have a plentiful supply of good immersions, and then faithful flat descent gives the rest. We bring the above ideas down to earth as follows:

Let $x \in X$ be a point closed in its fiber over Y . Let

- $y = f(x)$, $k = \mathcal{O}_{Y,y}/\mathfrak{m}_y$,
- $X_k = X \times_Y \text{Spec } k$, $\bar{x} \in X_k$ the closed point corresponding to $x \in X$,
- $A = \hat{\mathcal{O}}_{Y,y}$, $T = \text{Spec } A$,
- $u : T \rightarrow Y$ the natural map.

The map $u : T \rightarrow Y$ induces the diagram in 2.2.4 as well as a “compactified” version of that diagram

$$\begin{array}{ccccc}
 \bar{S}' & \xrightarrow{\bar{h}'} & \bar{S} & & \\
 \bar{v}' \searrow & & \swarrow \bar{v} & & \\
 \bar{X}' & \xrightarrow{\bar{g}'} & X & & \\
 \bar{f}' \downarrow & & \downarrow \bar{f} & & \\
 Y' & \xrightarrow{g} & Y & & \\
 u' \nearrow & & \nwarrow u & & \\
 T' & \xrightarrow{h} & T & &
 \end{array}$$

with \bar{f}' , \bar{f}'_T , \bar{g}' , \bar{h}' , \bar{v} and \bar{v}' being the compactifications of f' , f'_T , g' , h' , v and v' induced by the compactification \bar{f} of f .

Let $s \in S$ be the point corresponding to $\bar{x} \in X_k$. Since X_k is Cohen-Macaulay we can find an $\mathcal{O}_{X_k, \bar{x}}$ -sequence $\bar{t}_1 \dots, \bar{t}_r \in \mathfrak{m}_{\bar{x}}$. In an affine open neighborhood $U = \text{Spec } R$ of $s \in S$, we can lift $\bar{t}_1, \dots, \bar{t}_r$ to an R -sequence t_1, \dots, t_r . If Z is the closed subscheme of U defined by the t 's, then Z must be finite over T , for T is the spectrum of a *complete* local ring. Clearly $Z \xrightarrow{j} S$ is a good immersion for f_T . Now u and u' are *flat* and therefore $\theta_u^{\bar{f}}$ and $\theta_{u'}^{\bar{f}'}$ are isomorphisms. In view of Remark 2.2.3 we may consider $\theta_u^{\bar{f}}$ and $\theta_{u'}^{\bar{f}'}$ as identity maps², and we will do so. By Remark 2.2.4, the above identifications imply that

$$\theta_h^{\bar{f}_T} = \bar{v}'^* \theta_g^{\bar{f}}.$$

From our earlier arguments, for every $s' \in h^{-1}(s)$, $\theta_{h,s'}^{\bar{f}_T}$ is independent of \bar{f} and is an isomorphism. Now $g'^{-1}(x) = h'^{-1}(s)$. For $s' \in h'^{-1}(s)$, let x' denote the corresponding point in $g'^{-1}(x)$. We then have that the completion of $\bar{v}'^* \theta_g^{\bar{f}}$ at $s' \in S'$ is equal to the completion of $\theta_g^{\bar{f}}$ at $x' \in X'$. It follows (from the properties of $\theta_h^{\bar{f}_T}$) that $\theta_{g,x'}^{\bar{f}}$ is independent of \bar{f} and is an isomorphism for every $x' \in g'^{-1}(x)$. Since $x \in X$ was an arbitrary point closed in its fiber, therefore (as x varies) such x' are dense in X' . Parts (a) and (b) of Theorem 2.2.1 are immediate.

Remark 3.3.1. The Cohen-Macaulay hypothesis has been used in finding a good immersion $Z \hookrightarrow S$ over T . It is also used for getting the various local duality properties of res_Z , $g^\# \text{res}_{Z'}$ etc.

²by setting $\omega_{\bar{f}_T} = \bar{v}^* \omega_f$, $\omega_{\bar{f}'_T} = \bar{v}'^* \omega_{f'}$, $\int_{\bar{f}_T} = u^\# \int_{\bar{f}}$ and $\int_{\bar{f}'_T} = u'^\# \int_{\bar{f}'}$.

4. THE FUNDAMENTAL LOCAL ISOMORPHISM; ADJUNCTION

4.1. Sign convention for complexes: We follow the following (standard) sign conventions. These differ somewhat from the (non-standard) conventions in [6]. If A^\bullet and B^\bullet are complexes in an abelian category which admits a tensor product \otimes , then

- $\text{Hom}^\bullet(A^\bullet, B^\bullet)$ is the complex whose n th graded piece is

$$\text{Hom}^n(A^\bullet, B^\bullet) = \text{Hom}_{\text{gr}}(A^\bullet, B^\bullet[n])$$

and whose differential follows the rule

$$\begin{aligned} d^n f &= d_B \circ f - (-1)^n f \circ d_A \\ &= (-1)^{n+1} (f \circ d_A - d_{B[n]} \circ f). \end{aligned}$$

- $A^\bullet \otimes B^\bullet$ is the complex whose n -th piece is

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{p \in \mathbb{Z}} A^p \otimes B^{n-p}$$

and the differential is

$$d^n | A^p \otimes B^{n-p} = d_A^p \otimes 1 + (-1)^p \otimes d_B^{n-p}.$$

- We have a standard isomorphism

$$\theta_{ij} : A^\bullet[i] \otimes B^\bullet[j] \xrightarrow{\sim} (A^\bullet \otimes B^\bullet)[i+j]$$

which is “multiplication by $(-1)^{pj}$ ” on $A^{p+i} \otimes B^{q+j}$.

Now suppose R is a (commutative) ring. If P is a finitely generated projective module, we identify P with its double dual in the standard way. Let P^\bullet be complex of finitely generated projective modules over R . Then one checks (using the conventions above) that:

1. The complex $\tilde{P}^\bullet = \text{Hom}_R^\bullet(\text{Hom}_R^\bullet(P^\bullet, R), R)$ has as its differentials the *negatives* of the differentials of P^\bullet .
2. If Q^\bullet is the complex obtained from $\text{Hom}_R^\bullet(P^\bullet, R)$ by changing all the differentials to their negatives, then

$$P^\bullet = \text{Hom}_R^\bullet(Q^\bullet, R).$$

3. If M^\bullet is a complex of R modules, the natural isomorphism of R -modules

$$M^p \otimes_R \text{Hom}_R(P^s, R) \xrightarrow{\sim} \text{Hom}_R(P^s, M^p)$$

gives (without auxiliary signs) an isomorphism of complexes

$$M^\bullet \otimes_R \text{Hom}_R^\bullet(P^\bullet, R) \xrightarrow{\sim} \text{Hom}_R^\bullet(P^\bullet, M^\bullet).$$

Note the order in which the tensor product is taken.

4.2. The fundamental local isomorphism: Let R be a ring and I an ideal of R generated by an R -sequence $\mathbf{t} = (t_1, \dots, t_r)$. Let $B = R/I$, and $N_{B/R} = \bigwedge_B^r \text{Hom}_B(I/I^2, B) = \text{Hom}_B(\bigwedge_B^r I/I^2, B)$. For $t \in I$, let \bar{t} denote its image in I/I^2 . Now, $\bigwedge_B^r I/I^2$ is a free rank 1 B -module with $\bar{t}_1 \wedge \dots \wedge \bar{t}_r$ a generator. Denote by

$$\frac{1}{(t_1, \dots, t_r)} \in N_{B/R} \tag{4.1}$$

the dual generator (which sends $\bar{t}_1 \wedge \dots \wedge \bar{t}_r$ to $1 \in B$). Let $K^\bullet = K^\bullet(\mathbf{t}, R)$ denote the Koszul em cohomology complex on \mathbf{t} . There is, (from comments in the previous subsection) a complex of free R -modules C^\bullet such that

$$K^\bullet = \mathrm{Hom}_R^\bullet(C^\bullet, R).$$

In view of the sign conventions for Hom^\bullet , the complex C^\bullet is *not* the Koszul *homology* complex on \mathbf{t} , though it is canonically isomorphic to it, and as such it resolves the R -module B . It is well-known that K^\bullet resolves $N_{B/R}[-r]$ —the map $K^r = R \rightarrow N_{B/R}$ being the one which sends 1 to $1/(t_1, \dots, t_r)$. In the category $D^+(\mathrm{Mod}_R)$ we thus have two isomorphisms

$$\begin{aligned} B &\xrightarrow{\sim} C^\bullet \\ N_{B/R}[-r] &\xrightarrow{\sim} K^\bullet. \end{aligned}$$

For $M^\bullet \in D^+(\mathrm{Mod}_R)$ we have functorial isomorphisms

$$M^\bullet \overset{L}{\otimes}_R N_{B/R}[-r] \xrightarrow{\sim} M^\bullet \otimes_R K^\bullet \xrightarrow{\sim} \mathrm{Hom}_R^\bullet(C^\bullet, M^\bullet) \xrightarrow{\sim} R\mathrm{Hom}_R^\bullet(B, M^\bullet). \quad (4.2)$$

The resulting isomorphism between $M^\bullet \overset{L}{\otimes}_R N_{B/R}[-r]$ and $R\mathrm{Hom}_R^\bullet(B, M^\bullet)$ (obtained by composing the above isomorphisms together) is well known to be independent of the R -sequence generating I (even though the intermediate steps do depend on \mathbf{t}). This means we can globalize in the following way: Let $Z \xrightarrow{j} X$ be a (regular) closed immersion, i.e. the ideal \mathcal{I} of \mathcal{O}_X giving the immersion j is locally generated by an \mathcal{O}_X -sequence. Let \mathcal{N}_j denote the top wedge product of the normal bundle of the immersion $j : Z \hookrightarrow X$. Then in $D_{qc}^+(X)$ we have a functorial isomorphism—the *fundamental local isomorphism*

$$\mathcal{G}^\bullet \overset{L}{\otimes}_{\mathcal{O}_X} j_* \mathcal{N}_j[-r] \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{O}_X}^\bullet(j_* \mathcal{O}_Z, \mathcal{G}^\bullet). \quad (4.3)$$

4.3. Adjunction: Let $j : Z \hookrightarrow X$ be a closed immersion of schemes. We recall first the explicit description of duality for the map j . Let \mathcal{E}^\bullet be a bounded below complex of quasi-coherent, injective \mathcal{O}_X -modules, and \mathcal{J}^\bullet the injective \mathcal{O}_Z -complex satisfying $j_* \mathcal{J}^\bullet = \mathcal{H}om_{\mathcal{O}_X}^\bullet(j_* \mathcal{O}_Z, \mathcal{E}^\bullet)$. The adjoint properties of $\mathcal{H}om$ and \otimes gives, for any bounded below complex \mathcal{F}^\bullet functorial isomorphism \mathcal{O}_Z -modules, $\mathrm{Hom}_{\mathcal{O}_Z}^\bullet(\mathcal{F}^\bullet, \mathcal{J}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}^\bullet(j_* \mathcal{F}^\bullet, \mathcal{E}^\bullet)$. Since $j_* \mathcal{J}^\bullet$ is a complex of injective \mathcal{O}_X modules, we have that $j^! \mathcal{E}^\bullet \simeq \mathcal{J}^\bullet$, and under this identification, the trace map $j_* j^! \mathcal{E}^\bullet \rightarrow ce^\bullet$ is the natural inclusion $j_* \mathcal{J}^\bullet = \mathcal{H}om_{\mathcal{O}_X}^\bullet(j_* \mathcal{O}_Z, \mathcal{E}^\bullet) \hookrightarrow \mathcal{E}^\bullet$.

Now suppose $j : Z \hookrightarrow X$ is a regular immersion. For $\mathcal{G}^\bullet \in D_{qc}^+(X)$, set $j^! \mathcal{G}^\bullet = Lj^* \mathcal{G}^\bullet \otimes_{\mathcal{O}_Z} \mathcal{N}_j[-r]$. In view of the fundamental local isomorphism (4.3) above and our description of duality for j we have the *adjunction isomorphism*:

$$j^! \xrightarrow{\sim} j^\dagger. \quad (4.4)$$

If $f : X \rightarrow Y$ is a finite type map such that $h = f \circ j : Z \rightarrow Y$ is *proper*, then we have a map

$$\tau_Z : Rh_* j^\dagger f^! \longrightarrow R_Z f_* \quad (4.5)$$

arising from isomorphism (4.3) and the fact that $R\mathcal{H}om_{\mathcal{O}_X}(j_* \mathcal{O}_Z, _)$ is a subfunctor of $R\Gamma_Z$ (and the fact that $Rh_* = Rf_* \circ j_*$). Note also that the isomorphism (4.4)

gives us

$$h^! \xrightarrow{\sim} j^! f^!. \quad (4.6)$$

We would like to explicate the map $Rh_* j^! f^! \rightarrow 1_{D_{qc}^+(Y)}$ arising from the trace map $T_h : Rh_* h^! \rightarrow 1_{D_{qc}^+(Y)}$ and the above isomorphism. To that end, let (ι, \bar{f}) be a compactification of f . For $\mathcal{G}^\bullet \in D_{qc}^+(Y)$ define

$$T'_h(\mathcal{G}^\bullet) : Rh_* j^! f^! \mathcal{G}^\bullet \longrightarrow \mathcal{G}^\bullet$$

by the composition

$$Rh_* j^! f^! \mathcal{G}^\bullet \xrightarrow{\tau_Z} R_Z f_* f^! \mathcal{G}^\bullet \xrightarrow{\sim} R_Z \bar{f}_* \bar{f}^! \mathcal{G}^\bullet \longrightarrow R \bar{f}_* \bar{f}^! \mathcal{G}^\bullet \xrightarrow{T_{\bar{f}}} \mathcal{G}^\bullet.$$

We now come to the main point of all these seemingly meaningless exercises

Proposition 4.3.1. (a) T'_h does not depend on the compactification (ι, \bar{f}) .
 (b) The composition

$$Rh_* h^! \xrightarrow{(4.6)} Rh_* j^! f^! \xrightarrow{T'_h} 1_{D_{qc}^+(Y)}$$

is the trace map T_h

Proof. Part (a) is proved in the exactly the same way in which Proposition 3.1.1 is proved. Part (b) follows from the identity $T_h = T_{\bar{f} \circ R \bar{f}_*}(T_{\iota_j})$ (the functors can be composed only because we have implicitly made the identification $h^! = (\iota j)^! \bar{f}^!$ in the usual manner.) \square

5. LOCAL DUALITY

For this section, we assume $f : X \rightarrow Y$ is a *separated* Cohen-Macaulay map of relative dimension r . We also assume that we are given a good immersion $j : Z \hookrightarrow X$ for f . Set $h = f \circ j : Z \rightarrow Y$.

Now, $h : Z \rightarrow Y$ is flat (see [5, 15.1.16] or [17, p.177, Theorem 22.6]). Since h is finite, it then follows that h is *Cohen-Macaulay of relative dimension 0*. This means $H^i(h^! \mathcal{O}_Y) = 0$ of $i \neq 0$. This gives a canonical isomorphism $h^! \mathcal{O}_Y \simeq H^0(h^! \mathcal{O}_Y)$. Using this in conjunction with (4.6) we conclude that

$$Lj^* \omega_f[r] \otimes \mathcal{N}_j[-r] = j^* \omega_f[r] \otimes \mathcal{N}_j[-r].$$

Now set (in a suggestive notation)

$$\omega_h = j^* \omega_f \otimes_{\mathcal{O}_Z} \mathcal{N}_j$$

and define (in another suggestive notation)

$$\int_h : h_* \omega_h \longrightarrow \mathcal{O}_Y$$

by the composition

$$h_* \omega_h \xrightarrow{\theta_{r,r}^{-1}} h_*(\omega_f[r] \otimes \mathcal{N}_j[-r]) \xrightarrow{T'_h} \mathcal{O}_Y.$$

Here $\theta_{r,r}$ is “multiplication by $(-1)^r$ ” (see the definition of the map θ_{ij} in subsection 4.1 of the previous section.) The map \int_h is *a-priori* a map in $D_{qc}^+(Y)$, but since the source and target are concentrated in degree 0, \int_h is a map in Y_{qc} . In the definition of \int_h we have implicitly used the equality $Rh_* = h_*$ (h is an affine map).

The integral has another description. Taking the zero-th cohomology of the map $\tau_Z(\mathcal{O}_Y) \circ h_* \theta_{r,r}$ we get a map

$$r_Z : \omega_h \rightarrow R_Z^r f_* \omega_f.$$

Then clearly

$$\int_h = \text{res}_Z \circ r_Z.$$

Proposition 5.0.2. *The pair (ω_h, \int_h) is a dualizing pair for the map $h : Z \rightarrow Y$.*

Proof. This is a consequence of the definition of ω_h , \int_h and Proposition, 4.3.1. \square

The following is a version of local duality

Proposition 5.0.3. *The pairing given by the composition*

$$h_* j^* \omega_f \otimes_{\mathcal{O}_Y} h_* \mathcal{N}_j \longrightarrow h_*(j^* \omega_f \otimes_Z \mathcal{N}_j) = h_* \omega_h \xrightarrow{\int_h} \mathcal{O}_Z$$

is a perfect pairing of the \mathcal{O}_Y modules $h_ j^* \omega_f$ and $h_* \mathcal{N}_j$.*

Proof. From the definition of a good immersion, we may assume without loss of generality, that $Y = \text{Spec } A$, $X = \text{Spec } R$, $Z = \text{Spec } B$ and $B = R/I$, where I is generated by an R -sequence t_1, \dots, t_r . Our intent (clearly!) is to work with rings and modules, and we use the following dictionary $\omega_h \longleftrightarrow \omega_{B/A}$, $\int_h \longleftrightarrow \int_{B/A}$, $\omega_f \longleftrightarrow \omega_{R/A}$ and $N_j \longleftrightarrow N_{B/R} = N$. We have to show that the composed arrow

$$(B \otimes_R \omega_{R/A}) \otimes N \longrightarrow \omega_{B/A} = (B \otimes_R \omega_{R/A}) \otimes N \xrightarrow{\int_{B/A}} A$$

gives a perfect pairing between the A -modules $B \otimes_R \omega_{R/A}$ and N .

Since B is flat and finite over A (i.e. B is a projective A -module), the composition

$$\text{Hom}_A(B, A) \otimes_A B \longrightarrow \text{Hom}_A(B, A) \otimes_B B = \text{Hom}_A(B, A) \xrightarrow{e} A \quad (5.1)$$

(e = “evaluation at 1”) is a perfect pairing of the A -modules $\text{Hom}_A(B, A)$ and B . We will relate this pairing to the pairing stated in the Proposition to reach the desired conclusion. We have a B -isomorphism

$$\begin{aligned} \varphi : N &\xrightarrow{\sim} B \\ 1/(t_1, \dots, t_r) &\mapsto 1. \end{aligned}$$

By the adjoint properties of Hom and \otimes , we see that the pair $(\text{Hom}_A(B, A), e)$ represents the functor $\text{Hom}_A(-, A)$ of B -modules. But so does the pair $(\omega_{B/A}, \int_{B/A})$ (for (ω_h, \int_h) is a dualizing pair). We therefore have an isomorphism

$$\tilde{\psi} : \omega_{B/A} = (B \otimes_R \omega_{R/A}) \otimes_B N \xrightarrow{\sim} \text{Hom}_A(B, A)$$

such that $e \circ \tilde{\psi} = \int_{B/A}$. Let

$$\psi : B \otimes_R \omega_{R/A} \xrightarrow{\sim} \text{Hom}_A(B, A)$$

be the B -isomorphism induced by $\tilde{\psi}$ and φ . Clearly $\tilde{\psi} = \psi \otimes_B \varphi$. We have a commutative diagram

$$\begin{array}{ccccccc} (B \otimes_R \omega_{R/A}) \otimes_A N & \longrightarrow & (B \otimes_R \omega_{R/A}) \otimes_B N & \xlongequal{\quad \omega_{B/A} \quad} & \xrightarrow{\int_{B/A}} & A & \\ \psi \otimes_A \varphi \downarrow \simeq & & \simeq \downarrow \psi \otimes_B \varphi & & \downarrow \tilde{\psi} & & \parallel \\ \mathrm{Hom}_A(B, A) \otimes_A B & \longrightarrow & \mathrm{Hom}_A(B, A) \otimes_B B & \xlongequal{\quad} & \mathrm{Hom}_A(B, A) & \xrightarrow{e} & A. \end{array}$$

The bottom row is (5.1) which is a perfect pairing. The Proposition follows. \square

5.1. Koszul and Čech complexes: This subsection is a ragbag collection of well-known results concerning the relationship between local cohomology, Koszul complexes and Čech complexes. The crucial facts we wish to recount are the isomorphisms (5.2), (5.3) and the commutative diagram (5.4). We take the trouble to put this together because of the sign occurring in (5.4) (this sign was unfortunately missed in the comments following [13, p.61, (7.2.1)]. See also [15, p.115, (3.4)], where the same sign error is perpetuated).

Let R be a Noetherian ring, $I \subset R$ an ideal generated by an R -sequence $\mathbf{t} = (t_1, \dots, t_r)$, $Z = \mathrm{Spec} R/I$, $U = X \setminus Z$, $U_i = \mathrm{Spec} R_{t_i}$, $i = 1, \dots, r$, and $\mathcal{U} = \{U_i\}$. The assumption that \mathbf{t} form an R -sequence is one of convenience (so that the normal bundle to Z in X makes sense), but not always needed.

For a sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_r)$, let $\mathbf{t}^\alpha = (t_1^{\alpha_1}, \dots, t_r^{\alpha_r})$, $B_\alpha = B/\mathbf{t}^\alpha R$, $N_\alpha = N_{B_\alpha/R}$. For an R -module M , we define (as is standard in commutative algebra)

$$\Gamma_I(M) := \varinjlim_{\alpha} \mathrm{Hom}_R(B_\alpha, M) = \bigcup_{\alpha} (0 :_M \mathbf{t}^\alpha) \subset M.$$

For R -modules M , \widehat{M} will denote the completion of M in the I -adic topology (recall that $I \subset R$ is the ideal defining $Z \hookrightarrow X$). Note that

$$\widehat{M} = \varprojlim_{\alpha} M/\mathbf{t}^\alpha M.$$

Let $H^i(\mathbf{t}^\alpha, M) = H^i(K^\bullet(\mathbf{t}^\alpha, M))$. The last two maps in the series of isomorphisms in (4.2) give

$$H^i(\mathbf{t}^\alpha, M) \xrightarrow{\sim} \mathrm{Ext}_R^i(B_\alpha, M)$$

giving the well known isomorphism

$$\varinjlim_{\alpha} H^i(\mathbf{t}^\alpha, M) \xrightarrow{\sim} H_I^r(M).$$

Let $K_\infty^\bullet = \varinjlim_{\alpha} K^\bullet(\mathbf{t}^\alpha, R)$. Since $K_\infty^\bullet \otimes_R M = \varinjlim_{\alpha} K^\bullet(\mathbf{t}^\alpha, M)$ and since since all complexes in sight are concentrated in the degrees $0, \dots, r$, and since cohomology and tensor products commute with direct limits therefore the above considerations give isomorphisms

$$H_I^r(R) \otimes_R M \xrightarrow{\sim} H_I^r(M). \quad (5.2)$$

The same argument shows that if $R \rightarrow R'$ is a map of Noetherian rings with \mathbf{t} extending to an R' -sequence (strictly speaking this condition is not required for the isomorphism below) then

$$H_I^r(M) \otimes_R R' \xrightarrow{\sim} H_{IR'}^r(M \otimes_R R'). \quad (5.3)$$

Note that this isomorphism does not depend on the choice of the generators \mathbf{t} of I for the composition of the maps in (4.2) does not depend upon \mathbf{t} .

Recall that $K_\infty^p \otimes M = C^{p-1}(\mathcal{U}, \widetilde{M})$, where $C^\bullet(\mathcal{U}, \widetilde{M})$ is the Čech complex associated with the affine open cover \mathcal{U} of $U = X \setminus Z$. If δ and d are the differentials of $K_\infty^\bullet \otimes M$ and $C^\bullet(\mathcal{U}, \widetilde{M})$, then one checks that

$$\delta^p = d^{p-1}.$$

It follows that we have a surjective map (*equality* if $r > 1$!)

$$\varphi : \check{H}^{r-1} \rightarrow H^r(K_\infty^\bullet \otimes_R M).$$

One checks that the following diagram commutes:

$$\begin{array}{ccc} \check{H}^{r-1}(\mathcal{U}, \widetilde{M}) & \xrightarrow{\sim} & H^{r-1}(U, \widetilde{M}) \longrightarrow H_Z^r(X, \widetilde{M}) \\ (-1)^r \varphi \downarrow & & \parallel \\ H^r(K_\infty^\bullet \otimes M) & \xrightarrow{\sim} & H_I^r(M) \end{array} \quad (5.4)$$

where the map $H^{r-1}(U, \widetilde{M}) \longrightarrow H_Z^r(X, \widetilde{M})$ is the standard excision connecting map.³

Now, for $\alpha \leq \alpha'$ (the order being the lexicographic order), we have a map $N_\alpha \rightarrow N_{\alpha'}$ given by

$$\frac{1}{(t_1^{\alpha_1}, \dots, t_r^{\alpha_r})} \mapsto \frac{t_1^{\beta_1} \dots t_r^{\beta_r}}{(t_1^{\alpha'_1}, \dots, t_r^{\alpha'_r})}$$

where $\beta = \alpha' - \alpha$. This makes $\{N_\alpha\}$ into an inductive system. We have a commutative diagram

$$\begin{array}{ccccc} N_\alpha & \xrightarrow{\sim} & R[r] \otimes_R N_\alpha[-r] & \xrightarrow[\sim]{(4.2)} & R \operatorname{Hom}_R(B_\alpha, M) \\ \parallel & & \parallel & & \downarrow \\ N_\alpha & \xrightarrow{\theta_{r,r}^{-1}} & R[r] \otimes_R N_\alpha[-r] & \xrightarrow{(4.5)} & R\Gamma_I(M). \end{array}$$

Consider the bottom row. Taking the zeroth cohomology and applying \varinjlim_α (in either order) we get an isomorphism

$$\varinjlim_\alpha N_\alpha \xrightarrow{\sim} H_I^r(R) = H_I^r(\widehat{R}). \quad (5.5)$$

5.2. Proof of Proposition 3.2.1: It is clear that the statement of Proposition 3.2.1 is local on Y and that we can replace X by any open neighborhood of Z in X . We will, therefore, assume without loss of generality, that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} B$ etc. We use the notations used in the proof of Proposition 5.0.3 as well as those used in the last subsection. Granting Proposition 5.0.3, the proof we give is the standard proof given for e.g. in [13, p. 68, Theorem (7.4)]. We point out that unlike other statements of Local Duality for Cohen-Macaulay maps, we have no hypotheses on our base Y other than the Noetherian hypothesis.

³The reader is urged to experiment with $r = 1$. We point out that in our convention, connecting maps associated to a short exact sequence of complexes are obtained by the standard diagram chase without signs. This convention is dictated by the proof of [13, pp. 79–80, Lemma (8.6)] which is a crucial ingredient for the residue theorem for projective spaces of [*ibid*, Proposition (8.5)].

Let $\text{res}_I : H_I^r(\omega_{R/A}) \rightarrow A$ be the A -map corresponding to the residue map res_Z (in other words $\text{res}_I = \Gamma(Y \text{res}_Z)$). By Proposition 5.0.3 we have an isomorphism (of projective systems of R -modules)

$$\omega_{R/A}/\mathfrak{t}^\alpha \omega_{R/A} \xrightarrow{\sim} \text{Hom}_A(N_\alpha, A).$$

Taking projective limits, and using the isomorphism (5.5) we get

$$\widehat{\omega_{R/A}} \xrightarrow{\sim} \text{Hom}_A(H_I^r(R), A). \quad (5.6)$$

For any finitely generated \widehat{R} -module M we have a functorial isomorphism (see (5.2))

$$H_I^r(R) \otimes_{\widehat{R}} M \xrightarrow{\sim} H_I^r(M)$$

whence a functorial isomorphism

$$\text{Hom}_{\widehat{R}}(M, \text{Hom}_A(H_I^r(R), A)) \xrightarrow{\sim} \text{Hom}_A(H_I^r(M), A).$$

Using this and the isomorphism (5.6), we see that $(\widehat{\omega_{R/A}}, \text{res}_I)$ represents the functor $\text{Hom}_A(H_I^r(M), A)$ of finitely generated \widehat{R} -modules M . This completes the proof of Proposition 3.2.1.

6. BASE CHANGE FOR RESIDUES

6.1. Finite maps and base change: Suppose $f : X \rightarrow Y$ is a *finite* Cohen-Macaulay map (or what is the same thing—a finite flat map). Suppose further that we have a base change diagram.

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

Lemma 6.1.1. *The map $\theta_g^f : g'^* \omega_f \rightarrow \omega_{f'}$ is an isomorphism*

Proof. Without loss of generality, we may assume that $Y = \text{Spec } A$, $X = \text{Spec } B$, $Y' = \text{Spec } A'$ and $X' = \text{Spec } B'$ ($B' = B \otimes_A A'$). In view of Remark 2.2.3, we may choose any convenient dualizing pairs for f and f' . We have very simple descriptions of $(\omega_{B/A}, \int_{B/A})$ and $(\omega_{B'/A'}, \int_{B'/A'})$ in this case (the notations are self-explanatory). Since Hom and \otimes are adjoint functors,

$$\begin{aligned} \omega_{B/A} &= \text{Hom}_A(B, A) \\ \int_{B/A} : \omega_{B/A} &\rightarrow A \quad (\varphi \mapsto \varphi(1)) \end{aligned}$$

have the necessary dualizing property for the map f . Similarly $\omega_{B'/A'}$ can be identified with $\text{Hom}_{A'}(B', A')$ and $\int_{B'/A'}$ with “evaluation at 1”. The assertion that θ_g^f is an isomorphism reduces to checking that the natural map

$$\omega_{B/A} \otimes_A A' = \text{Hom}_A(B, A) \otimes_A A' \rightarrow \text{Hom}_{A'}(B \otimes_A A', A \otimes_A A') = \omega_{B'/A'} \quad (6.1)$$

is an isomorphism. Note that $\int_{B/A} \otimes_A A'$ maps to $\int_{B'/A'}$ under this map. To check that the map is an isomorphism, we only have to note that B is finite and flat over A , and hence it is a projective A -module. \square

Remark 6.1.1. It is worth pointing out that in this case (f finite), $g^* f_* = f'_* g'^*$, and hence we have an equality $g^\# \int_f = g^* \int_{f'}$.

6.2. Proof of Proposition 3.2.2. As usual, we may assume that $Y = \text{Spec } A$, $Y' = \text{Spec } A'$, $X = \text{Spec } R$, $X' = \text{Spec } R'$ and Z is defined by an ideal I of R generated by an R -sequence $\mathbf{t} = (t_1, \dots, t_r)$. Note that since R and $B = R/I$ are flat over A , therefore the extension of \mathbf{t} to R' is an R' -sequence (which is why Z' is a good immersion for f'). Let $Z' = \text{Spec } B'$. For a sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_r)$, let $B_\alpha = R/\mathbf{t}^\alpha R$ and $B'_\alpha = B_\alpha \otimes_A A'$. Let $N_\alpha = N_{B_\alpha/R}$ and $N'_\alpha = N_{B'_\alpha/R'}$. Note that $N'_\alpha = N_\alpha \otimes_A A'$.

If e and e' are the “evaluation at 1” maps of $\text{Hom}_A(B, A)$ and $\text{Hom}_{A'}(B', A')$, then we have a commutative diagram

$$\begin{array}{ccccc} (\text{Hom}_A(B, A) \otimes_A B) \otimes_A A' & \longrightarrow & \text{Hom}_A(B, A) \otimes_A A' & \xrightarrow{e \otimes A'} & A' \\ \simeq \downarrow & & \downarrow \simeq & & \parallel \\ \text{Hom}_{A'}(B', A') \otimes_A B' & \longrightarrow & \text{Hom}_{A'}(B', A') & \xrightarrow{e'} & A' \end{array}$$

where the vertical isomorphisms are as in (6.1). This translates to the statement that the composition

$$\begin{aligned} ((\omega_{R/A} \otimes_R B_\alpha) \otimes_A A') \otimes N'_\alpha &\longrightarrow ((\omega_{R/A} \otimes_A B_\alpha) \otimes_A A') \otimes_{B'_\alpha} N'_\alpha \\ &= (\omega_{R/A} \otimes_R N_\alpha) \otimes_A A' \\ &\xrightarrow{\int_{B_\alpha/A} \otimes A'} A' \end{aligned}$$

is a perfect pairing of the A' modules $(\omega_{R/A} \otimes_A B_\alpha) \otimes_A A'$ and N'_α . The proof is completed by taking the direct limit of this pairing over α as in Proposition 3.2.1, and noting that we have a commutative diagram

$$\begin{array}{ccc} (\omega_{R/A} \otimes_R N_\alpha) \otimes_A A' & \xlongequal{\quad} & (\omega_{R/A} \otimes_A A') \otimes_{R'} N'_\alpha \\ \downarrow & & \downarrow \\ H_I^r(\omega_{R/A}) \otimes_A A' & \longrightarrow & H_{IA'}^r(\omega_{R/A} \otimes_A A') \end{array}$$

in which the horizontal arrow at the bottom is (5.2) and the downward arrows are the ones defined by (5.5).

The proof of the proposition (especially the commutative diagram towards the end) together with Proposition 3.2.3 also gives the following corollary

Corollary 6.2.1. *Let $h = f \circ j : Z \rightarrow Y$, $h' = f' \circ j' : Z' \rightarrow Y'$ and let $g'' : Z' \rightarrow Z$ be the resulting map. Then*

(a) *Let $\theta = j'^* \theta_g^f \otimes 1_{\mathcal{N}_{j'}} : j'^*(g'^* \omega_f) \otimes_{\mathcal{O}_{Z'}} \mathcal{N}_{j'} \rightarrow j'^* \omega_{f'} \otimes_{\mathcal{O}_{Z'}} \mathcal{N}_{j'}$. Then*

$$\int_{h'} \circ h'_* \theta = g^* \int_h$$

where we are using the equalities

$$h'_*(j'^*(g'^*\omega_f) \otimes_{\mathcal{O}_{Z'}} \mathcal{N}_{j'}) = h'_*(g''^*(j^*\omega_f)) \otimes_{\mathcal{O}_{Z'}} \mathcal{N}_{j'} = g^*h_*(j^*\omega_f \otimes_{\mathcal{O}_Z} \mathcal{N}_j)$$

$$(b) \text{ } \text{res}_{Z'} \circ R^r f'_*(\theta_g^f) = g^\# \text{res}_Z.$$

6.3. Proof of Proposition 3.2.3: We point out that the “canonical” base change isomorphism $g^*R_Z f_* \simeq R_{Z'} f'_* g'^*$ referred to in the Proposition arises from globalizing the isomorphism (5.2) (see the comments below the isomorphism, allowing just such a globalization). So the proof of the Proposition reduces to the case where $Y = \text{Spec } A$, $Y' = \text{Spec } A'$, and Z is contained in an affine open subset $U = \text{Spec } R$ of X , and given by the vanishing of an R -sequence \mathbf{t} . The proof follows from (a) the diagram (5.4), (b) [15, pp. 79–80, (8.6)], (c) the definition of the isomorphism in (5.2) and (d) the fact that the base change map for higher direct images is compatible with its Čech cohomology version.

Remark 6.3.1. At this point we have proved completely parts (a) and (b) of Theorem 2.2.1 and hence also part (a) of Theorem 2.2.2. Consequently, the assertion in Remark 2.2.4 remains true even if f is not proper (or even separated), but under the added hypothesis that f is Cohen-Macaulay. This seems by locally compactifying f in an equidimensional (and possibly non Cohen-Macaulay) way.

7. THE SMOOTH CASE

We now prove part (c) of Theorem 2.2.1. This will complete the proof of Theorems 2.2.1 and 2.2.2. We first give a quick review of proof of the Verdier isomorphism (2.1).

7.1. Verdier again: . Suppose $f : X \rightarrow Y$ is separated and Cohen-Macaulay and $j : Z \hookrightarrow X$ is a good immersion for f , such that $h = f \circ j$ is an isomorphism⁴. If φ is the natural map $\mathcal{O}_Y \xrightarrow{\sim} h_* \mathcal{O}_Z$, then φ is an isomorphism and clearly $(\mathcal{O}_Z, \varphi^{-1})$ is a dualizing pair for h . This means that the map $\int_h : h_*(j^* \omega_f \otimes_{\mathcal{O}_Z} \mathcal{N}_j) \rightarrow \mathcal{O}_Y$ is an isomorphism, and this induces (via h_*^{-1}) an isomorphism

$$j^* \omega_f \otimes_{\mathcal{O}_Z} \mathcal{N}_j \xrightarrow{\sim} \mathcal{O}_Z. \quad (7.1)$$

Now suppose f is separated and smooth, and $P = X \times_Y X$, p_1 and p_2 the two projection maps $P \rightarrow X$, and $\delta : X \hookrightarrow P$ the diagonal map. Then δ is a good immersion for p_1 and (7.1) immediately gives us an isomorphism

$$u_f : \delta^* \omega_{p_1} \otimes_{\mathcal{O}_X} \mathcal{N}_\delta \xrightarrow{\sim} \mathcal{O}_X.$$

We have already shown that $\theta_f^f : p_2^* \omega_f \rightarrow \omega_{p_1}$ is an isomorphism. Plugging this into the isomorphism u_f above, and using the fact that $\delta^* p_2^*$ is the identity map on X_{qc} , we get an isomorphism

$$v_f : \omega_f \otimes_{\mathcal{O}_X} \mathcal{N}_\delta \xrightarrow{\sim} \mathcal{O}_X.$$

Tensoring both sides by Ω_f^r and using the fact that $\mathcal{N}_\delta^{-1} = \Omega_f^r$, we get Verdier’s isomorphism

$$\omega_f \xrightarrow{\sim} \Omega_f^r.$$

⁴This is tantamount to saying that f is smooth in a neighborhood of Z

7.2. Base change for smooth maps: Consider the situation in part (c) of Theorem 2.2.1. Let $P = X \times_Y X$ and $P' = X' \times_{Y'} X'$. We then have a commutative diagram

$$\begin{array}{ccccc}
 P' & \xrightarrow{p'_2} & X' & & \\
 \downarrow p'_1 & \searrow g'' & \downarrow g' & & \downarrow f' \\
 & P & \xrightarrow{p_2} & X & \\
 & \downarrow p_1 & & \downarrow f & \\
 & X' & \xrightarrow{f} & Y & \\
 & \uparrow g' & & \uparrow g & \\
 X' & \xrightarrow{f'} & Y' & &
 \end{array}$$

in which the outer square, the inner square, and the four trapeziums squeezed between them are all cartesian. According to Remark 6.3.1 the conclusions of Remark 2.2.4 apply in this case also. Therefore the diagram

$$\begin{array}{ccccc}
 p'_2{}^* g'^* \omega_f & \xrightarrow{p'_2{}^* \theta_g^f} & p'_2{}^* \omega_{f'} & \xrightarrow{\theta_{f'}^{f'}} & \omega_{p'_1} \\
 \parallel & & & & \parallel \\
 g''^* p_2{}^* \omega_f & \xrightarrow{g''^* \theta_f^f} & g''^* \omega_{p_1} & \xrightarrow{\theta_{p_1}^{p'_1}} & \omega_{p'_1}
 \end{array} \tag{7.2}$$

commutes. Let $\delta : X \rightarrow P$ and $\delta' : X' \rightarrow P'$ be the respective diagonal maps. Note that we have an amalgamation of cartesian diagrams:

$$\begin{array}{ccc}
 X' & \xrightarrow{g'} & X \\
 \delta' \downarrow & & \downarrow \delta \\
 P' & \xrightarrow{g''} & P \\
 p'_1 \downarrow & & \downarrow p_1 \\
 X' & \xrightarrow{g'} & X
 \end{array}$$

Let $\mathcal{N} = \mathcal{N}_\delta$ and $\mathcal{N}' = \mathcal{N}'_{\delta'}$. Corollary 6.2.1 (a) gives us a commutative diagram

$$\begin{array}{ccc}
 \delta'^*(g''^* \omega_{p_1}) \otimes \mathcal{N}' & \xlongequal{\quad} & g'^*(\delta^* \omega_{p_1} \otimes \mathcal{N}) \\
 \downarrow \text{via } \theta_{g'}^{p'_1} & & \downarrow \simeq g'^* u_f \\
 \delta'^*(\omega_{p'_1}) \otimes \mathcal{N}' & \xrightarrow[\sim]{u_{f'}} & \mathcal{O}_{Y'}.
 \end{array}$$

where u_f and $u_{f'}$ are the maps in subsection 7.1. Applying δ^* to diagram (7.2) and using the equality $\delta'^* g''^* = g'^* \delta^*$, we get a commutative diagram

$$\begin{array}{ccccc}
 g'^* \omega_f & \xrightarrow{\theta_g^f} & \omega_{f'} & \xrightarrow{\delta'^* \theta_{f'}^{f'}} & \delta'^* \omega_{p_1} \\
 \parallel & & & & \uparrow \delta'^* \theta_{g'}^{p_1} \\
 g'^* \omega_f & \xrightarrow{g'^* \delta^* \theta_f^f} & g'^* \delta^* \omega_{p_1} & \xlongequal{\quad} & \delta'^* g''^* \omega_{p_1}
 \end{array}$$

Note that the identifications $g'^* \mathcal{N} = \mathcal{N}'$ and $g'^* \Omega_f^r = \Omega_{f'}^r$ are compatible. Now put the above two diagrams together to get a commutative diagram

$$\begin{array}{ccc}
 g'^* \omega_f \otimes \mathcal{N}' & \xlongequal{\quad} & g'^* (\omega_f \otimes \mathcal{N}) \\
 \text{via } \theta_g^f \downarrow & & \downarrow \simeq g'^* v_f \\
 \omega_{f'} \otimes \mathcal{N}' & \xrightarrow[\sim]{v_{f'}} & \mathcal{O}_{Y'}.
 \end{array}$$

where v_f and $v_{f'}$ are as in subsection 7.1. Part (c) of Theorem 2.2.1 is immediate.

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